Physics 216 Mathematical Physics
Quiz 1, October 17, 2016, Time: 90 minutes

Solve 8 problems out of 10 (12.5 points for each)

1. Let $\{\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3\}$ be a basis of a three dimensional vector space V which are mapped by the operator T to the vectors $\{\overrightarrow{e}'_1, \overrightarrow{e}'_2, \overrightarrow{e}'_3\}$ defined by$T\overrightarrow{e}_i = \sum_{j=1}^{3} T_{ji}\overrightarrow{e}_j$ such that $\overrightarrow{e}'_1 = \overrightarrow{e}_1 + \overrightarrow{e}_3$, $\overrightarrow{e}'_2 = 2\overrightarrow{e}_1 + \overrightarrow{e}_2$, $\overrightarrow{e}'_3 = 3\overrightarrow{e}_2 + \overrightarrow{e}_3$. Find the matrix representation of T with respect to the basis $\{\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3\}$ and show that the set $\{\overrightarrow{e}'_1, \overrightarrow{e}'_2, \overrightarrow{e}'_3\}$ is also a basis.

2. Consider the two vectors $\overrightarrow{u} = (1, 3-i, 2+i)$ and $\overrightarrow{v} = (3, 1+i, 2i)$ in the vector space $\mathbb{C}^3$. Find $(\overrightarrow{u}, \overrightarrow{v})$, $(\overrightarrow{v}, \overrightarrow{u})$, $||\overrightarrow{u}||$, $||\overrightarrow{v}||$, $|(\overrightarrow{u}, \overrightarrow{v})|$. Verify that $\|(\overrightarrow{u}, \overrightarrow{v})\| \leq ||\overrightarrow{u}||\ ||\overrightarrow{v}||$.

3. Let $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$. Find $\det A$ and list the cofactors of the 9 elements of A to deduce $\operatorname{adj}(A)$ the classical adjoint matrix of A and the inverse matrix A^{-1}.

4. Consider the Hermitian matrix $A = \begin{pmatrix} 1 & 2-i\sqrt{2} \\ 2+i\sqrt{2} & 2 \end{pmatrix}$. Find the eigenvalues and normalized eigenvectors of A. Construct the diagonalizing matrix P.

5. Consider the operators $L_1 = -i\left(x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2}\right)$, $L_2 = -i\left(x^3\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^3}\right)$, $L_1 = -i\left(x^1\frac{\partial}{\partial x^2} - x^3\frac{\partial}{\partial x^1}\right)$. Show that the commutaor $[L_1, L_2]\ f\left(x^1, x^2, x^3\right) = iL_3 f\left(x^1, x^2, x^3\right)$.

6. Prove the identity$\overrightarrow{L} = -i\overrightarrow{r} \times \overrightarrow{\nabla} = i\left(\widehat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \widehat{\varphi}\frac{\partial}{\partial\varphi}\right)$, where $\overrightarrow{L}$ is the angular momentum. Evaluate $-\frac{i}{r^2}\overrightarrow{r} \times \overrightarrow{L}$ to deduce that $\overrightarrow{\nabla} = \widehat{r}\frac{\partial}{\partial r} - \frac{i}{r^2}\overrightarrow{r} \times \overrightarrow{L}$.

7. Let $\overrightarrow{V} = \frac{x}{\sqrt{x^2-y^2}}\overrightarrow{i} + \frac{y}{\sqrt{x^2-y^2}}\overrightarrow{j}$, $x > y$. Show that $\overrightarrow{V}$ is a conservative field and thus could be written as $\overrightarrow{\nabla}\phi$. Evaluate the line integral $\int_C \overrightarrow{V} \cdot \overrightarrow{dr}$ where C is a curve in the xy plane connecting the points $(1, 0)$ and (x, y). Deduce from this the function $\phi(x, y)$. Hint: Use $x = r\cosh\theta$, $y = r\sinh\theta$.

8. Let $\overrightarrow{V} = xy\overrightarrow{i} + z^2\overrightarrow{j} + 2yz\overrightarrow{k}$. Evaluate the surface integral $\oint_S \overrightarrow{V}.d\overrightarrow{a}$ where S is the surface of a cube $0 \leq x, y, z \leq 1$. Verify Gauss' law by evaluating the volume integral $\int_V \overrightarrow{\nabla} \cdot \overrightarrow{V}\, dxdydz$ where V is the volume bounded by the surface S.

9. Let $\overrightarrow{V} = y^2\overrightarrow{i} + xy\overrightarrow{j} + xz\overrightarrow{k}$. Evaluate the surface integral $\int_S \left(\overrightarrow{\nabla} \times \overrightarrow{V}\right) \cdot d\overrightarrow{a}$ where S is the upper hemisphere with equation $z = R^2 - x^2 - y^2$, $z \geq 0$ bounded by the circle C defined by $x^2 + y^2 = R^2$. Verify Stokes' theorem by evaluating the line integral $\oint_C \overrightarrow{V} \cdot d\overrightarrow{r}$. Hint: Write $d\overrightarrow{a} = R^2\sin\theta d\theta d\varphi\widehat{r}$ in spherical coordinates.

10. Verify the vector relation $\overrightarrow{\nabla} \cdot \left(\overrightarrow{A} \times \overrightarrow{B}\right) = -\overrightarrow{A} \cdot \left(\overrightarrow{\nabla} \times \overrightarrow{B}\right) + \overrightarrow{B} \cdot \left(\overrightarrow{\nabla} \times \overrightarrow{A}\right)$.